CONTEXTUALITY OF APPROXIMATE MEASUREMENTS

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Abstract

The claim of Meyer, Kent and Clifton (MKC) that finite precision measurement nullifies the Kochen-Specker theorem is criticised. It is argued that, although MKC have nullified the Kochen-Specker theorem strictly so-called, there are other, related propositions which are not nullified. The argument given is an elaboration of some of Mermin's critical remarks. Although MKC allow for the fact that the observables to be measured cannot be precisely specified, they continue to assume that the observables which are actually measured are strictly commuting. As Mermin points out, this assumption is unjustified. Consequently, the analysis of MKC is incomplete. To make it complete one needs to investigate the predictions their models make regarding approximate joint measurements of non-commuting observables. Such an investigation is carried out, using methods previously developed in connection with approximate joint measurements of position and momentum. It is shown that a form of contextuality then re-emerges.

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1. Introduction

In a recent series of papers Meyer [1], Kent [2] and Clifton and Kent [3] (to whom we will subsequently refer as MKC) claim to have "nullified" the Kochen-Specker theorem [4, 5, 6, 7]. They infer that "there is no truly compelling argument establishing that non-relativistic quantum mechanics describes classically inexplicable physics" [3]. They suggest that this may have significant consequences for quantum information theory and quantum computing.

The purpose of this paper is to criticize MKC's conclusions. It is true that MKC have circumvented the *particular kind* of non-classicality which features in the Kochen-Specker theorem. It is also true that in doing so they have significantly deepened our understanding of the conceptual implications of quantum mechanics. However, when it comes to the central question, as to whether non-relativistic quantum mechanics is classically explicable, it appears to us that a closer examination of their models leads to a different conclusion. We will argue that, although MKC have nullified the Kochen-Specker theorem *strictly so-called*, there are other, related propositions which are not nullified. The argument we will give is a development of some of the points made in Mermin's critique [8] (for other critical comments see Havlicek *et al* [9], Cabello [10] and Basu *et al* [11]).

As Mermin points out, MKC's analysis of finite precision measurements is not entirely adequate. MKC only consider one source of non-ideality: namely, the non-ideality which is due to inaccuracies in the specification of the observables to be measured. In every other respect the measurements they consider[1] are perfectly ideal (using the word "ideal" in the sense defined in Section 2). Such measurements might be rather better described (following Mermin) as ideal measurements which are not precisely specified. Consequently, the analysis of MKC is incomplete. In order to make it complete one needs to extend the analysis to the case of measurements which are not ideal *in any respect*: not ideal in respect of the target observables, which the apparatus is intended to measure; and not ideal in respect of any other observables either. The purpose of this paper is to present such an extended analysis. In the first part of the paper we give a more comprehensive account of approximate quantum mechanical measurements (based on ideas previously presented in Appleby [12, 13]). In the second part we apply these results to the MKC models.

It is important to distinguish the specific, technical result proved by Kochen and Specker, and the essential point of their argument. By the "essential point" we mean the proposition that quantum mechanics (whether relativistic or not) is inconsistent with classical conceptions of physical reality.

In the theories of classical physics it was tacitly assumed

1. To each observable quantity characterising a system there corresponds an objective physical quantity, which has a determinate value at every instant.
2. An ideal, perfectly precise measurement gives, *with certainty*, a value which *exactly coincides* with the value which the quantity being measured objectively did possess, immediately before the measurement process was initiated.

Of course, real laboratory measurements are not perfectly precise; and this fact was acknowledged in classical physics, just as it is in quantum physics. Consequently, the above propositions ought to be supplemented:

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[1] In the main part of their argument. The part of their argument which concerns (in their terminology) "positive operator measurements" will be discussed below (see Section 8).

3. A non-ideal, approximate measurement gives, *with high probability*, a value which is *close* to the value which the quantity being measured objectively did possess, immediately before the measurement process was initiated.

We will refer to these three propositions collectively as the principle of accessible objective values, or the AOV principle for short. Of course, if the AOV principle is not true, it does not necessarily follow that objective values do not exist. However, if the postulated objective values are typically quite different from the values obtained by measurement, then it is difficult to see what is achieved by assuming them. Consequently, failure of the AOV principle can be taken (though need not necessarily be taken) to justify a positivistic view: on the grounds that "a wheel that can be turned though nothing else moves with it, is not part of the mechanism" (as Wittgenstein [14] succinctly put it, in a different context). This was (in essence) the perception which motivated the Copenhagen Interpretation.

The significance of the Kochen-Specker theorem is that it seems to provide a rigorous proof that the AOV principle is inconsistent with the predictions of quantum mechanics. Kochen and Specker show that, if it is possible to make joint, ideal measurements of any set of commuting observables then, in a hidden variables theory, the result of making an ideal measurement of one observable must, in general, depend on which other commuting observables are jointly and ideally measured with it. This property is not consistent with clause 2 of the AOV principle.

The weakness in Kochen and Specker's argument was identified by MKC, who noted that the observables to be measured cannot be specified with perfect precision. Consequently, in an experiment which is intended to measure one set of commuting observables, the possibility cannot be excluded that what is actually measured is another, slightly different set of commuting observables. MKC use this freedom to construct a hidden variables theory which *does* satisfy clause 2 of the AOV principle. It should be noted that the theory they construct is not strictly equivalent to standard quantum mechanics (because they postulate that an observable can only be measured if it belongs to a particular, proper subset of the set of all self-adjoint operators). Consequently, they have not shown that clause 2 of the AOV principle is consistent with all the predictions of quantum mechanics (*i.e.*, they have not *refuted* the Kochen-Specker theorem). On the other hand, they have shown that this clause is consistent with the predictions of quantum mechanics in so far as these are empirically verifiable [*i.e.*, they have *nullified* the Kochen-Specker theorem (strictly so-called)].

However, just as MKC have noted a significant weakness in the argument of Kochen and Specker, so in turn Mermin [8] has noted a significant weakness in theirs. Although MKC allow for the fact that the observables which are actually measured may be slightly and uncontrollably different from the target observables, which the experiment is intended to measure, they nevertheless follow Kochen and Specker in assuming that the observables which are actually measured still are strictly commuting. But, as Mermin points out, MKC's own assumptions suggest that the observables which are actually measured will almost certainly *not* be strictly commuting. Also (and, as it turns out, closely connected with this point) one may ask: do the MKC models satisfy clause 3 of the AOV principle? After all, if one accepts MKC's starting point, that perfect precision is practically unattainable, then clause 3 of the AOV principle, relating as it does to approximate measurements, must be the one which really matters. Clause 2, by contrast, relating as it does to ideal measurements—which is to say practically unrealizable measurements—must be regarded as being of negligible importance. Yet clause 2 is the only one on which their argument bears.

In order to appreciate the force of Mermin's point it will be helpful to specialise to the standard example of a spin-1 particle, with angular momentum $\hat{\mathbf{L}}$. Kochen and Specker consider joint measurements of the three commuting projectors $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$ for $r = 1, 2, 3$ where $\mathbf{e}_r$ is any orthonormal triad in $\mathbb{R}^3$. A schematic arrangement for performing such a measurement using three separate analyzers is illustrated in Fig. 1. MKC correctly observe that, in such an arrangement, it would not

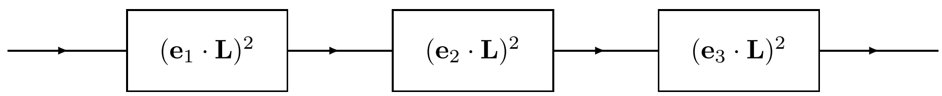

Figure 1. Schematic arrangement for jointly measuring the observables $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$. The system passes through a succession of analysers, which measure each observable in turn.

practically be possible to align the analyzers precisely along the three directions $\mathbf{e}_r$. In practice one would expect there to be some uncontrollable errors, so that what are actually measured are the projections $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$, where the triad $\mathbf{e}'_r$ is close, but not exactly coincident with the triad $\mathbf{e}_r$. Nevetheless, MKC assume that the triad $\mathbf{e}'_r$ is precisely orthonormal. Yet it seems clear that, given that the errors are random and uncontrollable, and given that the analyzers are separate instruments, one would typically expect there to be some slight departures from strict orthogonality. At the least, there are no evident grounds for assuming the contrary.

MKC assume that the triad $\mathbf{e}'_r$ must be exactly orthonormal because they follow Kochen and Specker in relying on the principle that it is only sets of commuting observables which can jointly be measured with perfect accuracy. However, if one relaxes the requirement that the measurements be perfectly accurate, then this principle is no longer valid. There is now an extensive literature on the subject of joint, inexact measurements of non-commuting observables. To date, the topics which have received most attention are joint measurements of position and momentum [12, 13, 15, 16, 17], and joint measurements of the components of spin [18, 19]. For recent reviews, and additional references, the reader may consult Busch *et al* [20], and Leonhardt [21]. It should be stressed that recent advances in the field of quantum optics mean that such measurements can now be realized, in the laboratory.

The purpose of this paper is to investigate the consequences of extending the analysis of MKC, so as to include approximate joint measurements of non-commuting observable.

The paper is in two main parts. The first part, comprising Sections 2–5, is concerned with the theory of approximate measurements. The discussion in these sections is based on ideas previously presented in Appleby [12, 13] (also see Appleby [16, 17, 19]). In our earlier papers we were concerned with the two special cases, of approximate joint measurements of position and momentum [12, 13, 16, 17], and approximate joint measurements of the components of spin [19]. In Sections 2 and 3 we show how the same methods can be used to analyse approximate measurements for any system having a finite dimensional state space (the extension to the case of a system having an infinite dimensional state space is straightforward,

but unnecessary for present purposes). In Section 4 we discuss Uffink's [22] criticisms of the description of approximate joint measurement processes which (unlike the approach taken in this paper) is based on the concept of an unsharp observable. In Section 5 we specialise the discussion to the case of approximate joint measurements of the operators $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$ introduced above, where $\hat{\mathbf{L}}$ is the angular momentum for a spin-1 system, and where the triad $\mathbf{e}'_r$ is not assumed to be orthonormal.

The account in Sections 2–5 is somewhat lengthy. This is because there are some subtleties, and serious potential confusions, which make it necessary to give a detailed discussion of the underlying concepts. Uffink [22] (also see Fleming [23]) has identified some obscurities in the theory of joint measurements of non-commuting observables as it is presented by (for example) Busch *et al* [20]. It so happens that these objections have a direct bearing on the questions addressed in this paper. One of the advantages of our approach is that Uffink's objections do not apply to it (the other advantage of our approach being that it leads to an improved [16] definition of measurement accuracy). The importance of this fact will appear in Section 8, where we consider the argument of Clifton and Kent [3] which (they claim) "rule[s] out falsifications of non-contextual models based on generalized observables, represented by POV measures".

In Section 6 we apply the concepts and methods developed in Sections 2–5 to joint measurements of the target observables $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$, under circumstances where the triad $\mathbf{e}_r$ is not precisely specified, so that the observables $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$ which are actually measured cannot be assumed to be precisely commuting. We show that, if the errors in the alignments of the vectors $\mathbf{e}_r$ are statistically independent, then it is possible to prove a modified version of the Kochen-Specker theorem (a Kochen-Specker theorem for approximate measurements, as it might be called), from which it follows that clause 3 of the AOV principle is not satisfied. This argument was originally inspired by some of the points made by Mermin [8] (although it appears to us that our formulation is considerably sharper than Mermin's: moreover, Mermin does not remark on the need to assume that the errors are independent).

The result proved in Section 6 shows that, if the alignment errors are independent, then the outcome of an approximate measurement must, in general, be strongly dependent on the particular manner in which the measurement is carried out. In other words, the theory must exhibit a kind of contextuality. However, it may be that there are theories of the MKC type for which the errors are not independent. This possibility is discussed in Section 7. We begin by remarking that, although it is conceivable that there exist theories of the type proposed by MKC for which the errors are not independent, and which do satisfy all three clauses of the AOV principle, it would not be straightforward actually to prove that this was the case. The distribution of errors is not a feature of the theory which one is free simply to postulate. In a complete theory it should be a consequence of the detailed dynamics of the interaction between the system, the measuring apparatus, and the environment. The models proposed by MKC are, as they stand, incomplete, since they do not include any dynamical postulate. In order to show that they satisfy clause 3 the AOV principle it would be necessary, first to specify the dynamical evolution of the hidden variables which characterise the interacting system+apparatus+environment composite, and then to work out the distribution of errors which this implies. Moreover, one would need to establish that the assumption of independence fails in just the way that is required for clause 3 of the AOV principle to be satisfied; and one would need to show that this is the case for every possible system, and every possible set of measurements. Such a program would constitute a highly non-trivial theoretical undertaking. We may therefore conclude, in the first place, that it remains an open question, whether there exists

a hidden variables theory satisfying clause 3 of the AOV principle. Whether or not this clause *can* be nullified, it has not been nullified *yet*.

In the second place it is to be observed that such a theory, if it could be constructed, would entail the existence of a delicately adjusted collaboration between the ostensibly random fluctuations in the different parts of a composite apparatus. In Section 7 we argue that this would itself represent a kind of contextuality: for it would mean that the fluctuations in each component of a complex apparatus were, in general, intricately and inescapably dependent on the overall experimental context in which that component was employed. In other words, a theory of this kind would not so much eliminate the phenomenon of contextuality, as shift the locus of the contextuality, from the system, onto the fluctuations in the measuring apparatus.

Our overall conclusion consequently is that, no matter how the theoretical postulates are adjusted, some kind of contextuality must appear somewhere.

Finally, in Section 8 we discuss Clifton and Kent's [3] theorem 2 which is intended to "rule out falsifications of non-contextual models based on generalized observables, represented by POV measures" (also see Kent [2]). Approximate joint measurements of non-commuting observables are most conveniently described using a POVM, and so it may at first sight seem that Clifton and Kent's theorem 2 contradicts the result proved in Section 6 of this paper. In fact, this is not the case, as we show in Section 8. The reason is connected with the point made in Section 4: namely, that although the concept of an approximate measurement involves the concept of a POVM, it does not involve the concept of a new kind of "generalized observable", distinct from the ordinary kind of observable which is represented by a self-adjoint operator. We go on to discuss some other difficulties which arise from the way in which Clifton and Kent use the concept of a generalized observable.

2. Approximate Measurements

The purpose of this section and the one following is to give a general characterisation of approximate measurement(s) performed on a system having a finite dimensional state space. The observables being measured may be commuting or non-commuting. Our approach is based on ideas previously presented in Appleby [12, 13], in connection with approximate joint measurements of position and momentum (also see Appleby [16, 17, 19]). As discussed in Section 4, our approach differs from the one taken by many other authors in that it makes no use of the concept of an unsharp observable. This will prove relevant in Section 8, when we discuss Clifton and Kent's [3] theorem 2. The basic physical ideas are described in this section. The mathematical elaboration in terms of POVM's is described in Section 3.

An approximate measurement is a measurement which is less than perfectly accurate. It follows, that in order properly to characterise an approximate measurement it is necessary first to arrive at a satisfactory, quantum mechanical concept of measurement accuracy. This is the problem to which we now turn. We begin by considering the accuracy of an imperfect measurement of a single observable. We then extend the discussion to the case of simultaneous, imperfect measurements of a set of several different observables (commuting or non-commuting).

The ordinary, intuitive concept of accuracy involves a comparison between the result of the measurement, and the original value which the quantity being measured did take, immediately before the measurement was carried out. In a quantum mechanical context this concept becomes problematic. The reason for this is the very feature of quantum mechanics which the Kochen-Specker theorem was intended to establish: namely, the fact that in quantum mechanics the concept of "the

original value of the observable being measured" is not always well-defined. Of course, one is free to make it well-defined, by taking a hidden variables approach. However, this way of arriving at a concept of quantum mechanical accuracy is not satisfactory because, quite apart from the fact that it compels one to favour a hidden variables approach over all the many alternatives, it makes the accuracy strongly dependent on which particular hidden variables theory is adopted. It is arbitrary, in other words. What one wants is a concept of accuracy which is (1) a natural generalization of the classical concept and (2) independent of the way in which the theory is interpreted (so that it is a feature of quantum mechanics *as such*, and not simply a feature of this or that particular interpretation). In the following we will present a solution to this problem.

Let us start with the standard, elementary textbook example of a measurement process. Consider a system, with finite dimensional state space $\mathcal{H}_{\mathrm{sy}}$, and an apparatus, with finite dimensional state space $\mathcal{H}_{\mathrm{ap}}$. Let $\hat{A}$ be a system observable acting on $\mathcal{H}_{\mathrm{sy}}$, and let $\hat{\alpha}$ be a pointer observable acting on $\mathcal{H}_{\mathrm{ap}}$. Suppose that $\hat{A}$ and $\hat{\alpha}$ have the same set of eigenvalues $\{a\}$, which for simplicity we will assume to be non-degenerate. Let $|a\rangle_{\mathrm{sy}}$ be the corresponding eigenvectors of the operator $\hat{A}$, and let $|a\rangle_{\mathrm{ap}}$ be the eigenvectors of $\hat{\alpha}$. Let $|\phi_0\rangle_{\mathrm{ap}}$ be the initial "zeroed" or "ready" state of the apparatus, and let $\sum_a c_a |a\rangle_{\mathrm{sy}}$ be the initial state of the system. We then obtain an idealised measurement process by postulating that the unitary evolution operator $\hat{U}$ describing the interaction between system and apparatus is such that

$$\hat{U}\left(\left(\sum_a c_a |a\rangle_{\mathrm{sy}}\right) \otimes |\phi_0\rangle_{\mathrm{ap}}\right) = \sum_a c_a \left(|a\rangle_{\mathrm{sy}} \otimes |a\rangle_{\mathrm{ap}}\right) \tag{1}$$

What makes this a measurement is the fact that it establishes a correlation between the system and pointer observables. What makes it ideal is the fact that the correlation is, in a certain sense, perfect. Specifically:

1. The measurement is retrodictively ideal in the sense that, if the system was initially in the eigenstate of $\hat{A}$ with eigenvalue a, then there is probability 1 that the recorded value of the pointer observable will also be a. Consequently, if the system was prepared in some unknown eigenstate of $\hat{A}$, the result of the measurement can be used to retrodict, with certainty, which particular eigenstate it was.
2. The measurement is predictively ideal in the sense that, if the pointer observable is recorded as having the value a immediately after the measurement, then one can predict, with probability 1, that a second, immediately subsequent retrodictively ideal measurement of $\hat{A}$ will give the same value a.

It is easily seen that these two properties, of retrodictive and predictive ideality, are independent. That is, there exist unitary evolution operators $\hat{U}$ describing processes which are retrodictively but not predictively ideal; and operators $\hat{U}$ describing processes which are predictively but not retrodictively ideal.

Practically speaking perfection is seldom, if ever attainable. Consequently, one does not expect a real measurement process to be either retrodictively or predictively ideal. A more realistic model of a measurement process is obtained if, instead of Eq. (1), we take the evolution to be described by

$$\hat{U}\left(\left(\sum_a c_a |a\rangle_{\mathrm{sy}}\right) \otimes |\phi_0\rangle_{\mathrm{ap}}\right) = \sum_a c_a \left(|a\rangle_{\mathrm{sy}} \otimes |a\rangle_{\mathrm{ap}}\right) + \sum_{a,b,d} c_a \epsilon_{a,bd} \left(|b\rangle_{\mathrm{sy}} \otimes |d\rangle_{\mathrm{ap}}\right) \tag{2}$$

where $|\epsilon_{a,bd}| \ll 1$ for all a, b, d. Of course, this model does not include all the complications which one might expect to find in a real measurement process. A complete

account should allow for the existence of other apparatus degrees of freedom, additional to $\hat{\alpha}$. It should also allow for the interaction with the environment [24], and for the fact that $\hat{A}$ and $\hat{\alpha}$ may not have exactly the same spectrum. However, the model just indicated has the merit of simplicity, and it will serve to illustrate the essential ideas.

If the coefficients $\epsilon_{a,bd}$ are sufficiently small, then the process described by Eq. (2) may be regarded as an approximate measurement of $\hat{A}$: for, corresponding to the properties 1 and 2 above, we have

3. The measurement is retrodictively good in the sense that, if the system was prepared in some unknown eigenstate of $\hat{A}$, then the result of the measurement can be used to retrodict, with a high degree of confidence, which particular eigenstate it was.
4. The measurement is predictively good in the sense that, if the pointer observable is recorded as having the value a immediately after the measurement, then one can predict, with probability close to 1, that a second, immediately subsequent retrodictively ideal measurement of $\hat{A}$ will give the same value a.

We next show how it is possible to quantify the degree of accuracy of the measurement. Define

$$\hat{A}_{\mathrm{f}} = \hat{U}^{\dagger}\hat{A}\hat{U}$$
$$\hat{\alpha}_{\mathrm{f}} = \hat{U}^{\dagger}\hat{\alpha}\hat{U}$$

$\hat{A}_{\mathrm{f}}$, $\hat{\alpha}_{\mathrm{f}}$ are the final Heisenberg picture observables, defined at the moment the measurement interaction is completed. Let $\hat{A}_{\mathrm{i}} = \hat{A}$ denote the initial Heisenberg picture system observable, defined at the moment the measurement interaction begins. Define the retrodictive error operator $\hat{\epsilon}_{\mathrm{i}}$ and predictive error operator $\hat{\epsilon}_{\mathrm{f}}$ by

$$\hat{\epsilon}_{\mathrm{i}} = \hat{\alpha}_{\mathrm{f}} - \hat{A}_{\mathrm{i}} \tag{3}$$
$$\hat{\epsilon}_{\mathrm{f}} = \hat{\alpha}_{\mathrm{f}} - \hat{A}_{\mathrm{f}} \tag{4}$$

Let $\mathbb{S}_{\mathrm{sy}}$ denote the unit sphere $\subset \mathcal{H}_{\mathrm{sy}}$. Following the discussion in Appleby [13] we now define the maximal rms error of retrodiction, $\Delta_{\mathrm{ei}}A$ by

$$\Delta_{\mathrm{ei}}A = \left(\sup_{\psi\in\mathbb{S}_{\mathrm{sy}}}\left(\left\langle\psi\otimes\phi_0\right|\hat{\epsilon}_{\mathrm{i}}^2\left|\psi\otimes\phi_0\right\rangle\right)\right)^{\frac{1}{2}} \tag{5}$$

and the maximal rms error of prediction, $\Delta_{\mathrm{ef}}A$ by

$$\Delta_{\mathrm{ef}}A = \left(\sup_{\psi\in\mathbb{S}_{\mathrm{sy}}}\left(\left\langle\psi\otimes\phi_0\right|\hat{\epsilon}_{\mathrm{f}}^2\left|\psi\otimes\phi_0\right\rangle\right)\right)^{\frac{1}{2}} \tag{6}$$

Of these two quantities the predictive error $\Delta_{\mathrm{ef}}A$ is the easier to interpret because $\hat{\epsilon}_{\mathrm{f}}$ (unlike $\hat{\epsilon}_{\mathrm{i}}$) connects Heisenberg picture observables defined at the same instant of time. Let $|\psi\rangle \in \mathcal{H}_{\mathrm{sy}}$ be the (normalised) initial system state. Then, reverting to the Schrödinger picture,

$$\left(\left\langle\psi\otimes\phi_0\right|\hat{U}^{\dagger}(\hat{\alpha}-\hat{A})^2\hat{U}\left|\psi\otimes\phi_0\right\rangle\right)^{\frac{1}{2}} \leq \Delta_{\mathrm{ef}}A$$

from which we see that, the smaller $\Delta_{\mathrm{ef}}A$, the more closely the result of a second, immediately subsequent, retrodictively ideal measurement of $\hat{A}$ may be expected to approximate the result of the (non-ideal) measurement under discussion. In particular, if $\Delta_{\mathrm{ef}}A = 0$, then the measurement is predictively ideal. It is not difficult to see that the condition $\Delta_{\mathrm{ef}}A = 0$ is in fact, not only sufficient, but also necessary for the measurement to be predictively ideal. This justifies the interpretation of $\Delta_{\mathrm{ef}}A$ as providing a quantitative indication of the degree of predictive accuracy.

Let us now consider the interpretation of the quantity $\Delta_{\mathrm{ei}} A$. Suppose, to begin with, that the initial system state $|\psi\rangle$ is an eigenstate of $\hat{A}$ with eigenvalue a. Then

$$\Bigl(\bigl\langle \psi\otimes\phi_0\bigr|\, \hat{U}^{\dagger}(\hat{\alpha}-a)^2\hat{U}\, \bigl|\psi\otimes\phi_0\bigr\rangle\Bigr)^{\frac{1}{2}} = \Bigl(\bigl\langle \psi\otimes\phi_0\bigr|\, (\hat{\alpha}_{\mathrm{f}}-a)^2\, \bigl|\psi\otimes\phi_0\bigr\rangle\Bigr)^{\frac{1}{2}} \leq \Delta_{\mathrm{ei}} A$$

from which we see that, the smaller $\Delta_{\mathrm{ei}} A$, the more closely the recorded value of the pointer observable may be expected to approximate a, and the more accurate the measurement is retrodictively. In particular, if $\Delta_{\mathrm{ei}} A = 0$, then the measurement is retrodictively ideal. It is not difficult to see that the condition $\Delta_{\mathrm{ei}} A = 0$ is in fact both necessary and sufficient for the measurement to be retrodictively ideal.

It is also possible to say something about the result of the measurement in the case when the initial system state $|\psi\rangle$ is not an eigenstate of $\hat{A}$. Let $\bar{A}$ and ΔA denote the initial state mean and uncertainty:

$$\begin{aligned} \bar{A} &= \langle\psi|\, \hat{A}\, |\psi\rangle \\ \Delta A &= \Bigl(\langle\psi|\, (\hat{A}-\bar{A})^2\, |\psi\rangle\Bigr)^{\frac{1}{2}} \end{aligned}$$

Then the spread of measured values about the initial state mean satisfies the inequality

$$\begin{aligned} \Bigl(\bigl\langle \psi\otimes\phi_0\bigr|\, (\hat{\alpha}_{\mathrm{f}}-\bar{A})^2\, \bigl|\psi\otimes\phi_0\bigr\rangle\Bigr)^{\frac{1}{2}} &\leq \Bigl(\bigl\langle \psi\otimes\phi_0\bigr|\, (\hat{\alpha}_{\mathrm{f}}-\hat{A}_{\mathrm{i}})^2\, \bigl|\psi\otimes\phi_0\bigr\rangle\Bigr)^{\frac{1}{2}} \\ &\qquad + \Bigl(\bigl\langle \psi\otimes\phi_0\bigr|\, (\hat{A}_{\mathrm{i}}-\bar{A})^2\, \bigl|\psi\otimes\phi_0\bigr\rangle\Bigr)^{\frac{1}{2}} \\ &\leq \Delta_{\mathrm{ei}} A + \Delta A \end{aligned} \tag{7}$$

We see from this that there are two components to the spread of measured values. ΔA represents the intrinsic uncertainty of the initial system state. $\Delta_{\mathrm{ei}} A$ represents an upper bound on the extrinsic uncertainty, attributable to the noise introduced by the measuring procedure.

These considerations justify the interpretation of $\Delta_{\mathrm{ei}} A$ as providing a quantitative indication of the degree of retrodictive accuracy.

Finally we note that the necessary and sufficient condition for the coefficients $\epsilon_{a,bd}$ in Eq. (2) all to be zero (so that the measurement is completely ideal) is that $\Delta_{\mathrm{ei}} A = \Delta_{\mathrm{ef}} A = 0$.

Let us now consider a joint measurement of several different observables. If the observables are mutually commuting then the above discussion generalises in the obvious way. However, the point which is important for the argument of this paper is that it also generalises, in a manner which is only slightly less obvious, to the case when the observables are *not* mutually commuting. It is true that one cannot make completely ideal joint measurements of a set of non-commuting observables. However, there is nothing to preclude one from making joint measurements which are only approximate.

Let $\hat{A}_1, \dots, \hat{A}_n$ be the non-commuting observables to be measured, acting on the system state space $\mathcal{H}_{\mathrm{sy}}$. Corresponding to these observables we introduce a set of n pointer observables $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ acting on the apparatus state space $\mathcal{H}_{\mathrm{ap}}$. We take it that the observables $\hat{\alpha}_1, \dots, \hat{\alpha}_n$, unlike the observables $\hat{A}_1, \dots, \hat{A}_n$, are mutually commuting. Consequently, their joint eigenvectors constitute an orthonormal basis for $\mathcal{H}_{\mathrm{ap}}$. Let $|a_1, \dots, a_n\rangle_{\mathrm{ap}}$ denote the joint eigenvector with eigenvalues $a_1, \dots, a_n$ (for the sake of simplicity we assume that the eigenstates are non-degenerate). As before, let $|\phi_0\rangle$ be the initial apparatus "zeroed" or "ready" state, and let $\hat{U} \colon \mathcal{H}_{\mathrm{sy}} \otimes \mathcal{H}_{\mathrm{ap}} \to \mathcal{H}_{\mathrm{sy}} \otimes \mathcal{H}_{\mathrm{ap}}$ be the unitary evolution operator describing the measurement interaction.

The fact that the observables $\hat{A}_1, \dots, \hat{A}_n$ are non-commuting means that we cannot choose a basis for $\mathcal{H}_{\text{sy}}$ which consists of their joint eigenvectors. However, we can choose, for each r separately, a basis which consists of eigenvectors just of $\hat{A}_r$. Let $|a, x\rangle_r$ be such a basis (where a denotes the eigenvalue, and the additional index x is to allow for possible degeneracies). We may then write

$$\hat{U}\left(\left(\sum_{a,x} c_{ax} \left| a, x \right\rangle_r \right) \otimes |\phi_0\rangle \right) = \sum_{b,y,d_1,\dots,d_n} c_{ax} f^{(r)}_{ax;by;d_1\dots d_n} \left(|b, y\rangle_r \otimes |d_1, \dots, d_n\rangle_{\text{ap}} \right)$$

for suitable coefficients $f^{(r)}_{ax;by;d_1\dots d_n}$. Suppose that, for all r, these coefficients have the property that $f^{(r)}_{ax;by;d_1\dots d_n}$ is small except when $a = b = d_r$. Then, comparing this equation with Eq. (2), we see that, for each r, the pointer $\hat{\alpha}_r$ provides an approximate measurement of the system observable $\hat{A}_r$. This situation may appropriately be described by saying that the process provides an approximate joint measurement of the set of observables $\hat{A}_1, \dots, \hat{A}_n$.

As in the case of approximate measurements of a single observable, we may obtain a quantitative indication of the accuracy by making use of the Heisenberg picture observables $\hat{A}_{r\text{f}} = \hat{U}^\dagger \hat{A}_r \hat{U}$, $\hat{\alpha}_{r\text{f}} = \hat{U}^\dagger \hat{\alpha}_r \hat{U}$, $\hat{A}_{r\text{i}} = \hat{A}_r$. By analogy with Eqs. (3) and (4) define

$$\hat{\epsilon}_{r\text{i}} = \hat{\alpha}_{r\text{f}} - \hat{A}_{r\text{i}}$$
$$\hat{\epsilon}_{r\text{f}} = \hat{\alpha}_{r\text{f}} - \hat{A}_{r\text{f}}$$

We then obtain n maximal rms errors of retrodiction

$$\Delta_{\text{ei}} A_r = \left(\sup_{\psi \in \mathcal{S}_{\text{sy}}} \left(\left\langle \psi \otimes \phi_0 \right| \hat{\epsilon}^2_{r\text{i}} \left| \psi \otimes \phi_0 \right\rangle \right) \right)^{\frac{1}{2}} \tag{8}$$

and n maximal rms errors of prediction

$$\Delta_{\text{ef}} A_r = \left(\sup_{\psi \in \mathcal{S}_{\text{sy}}} \left(\left\langle \psi \otimes \phi_0 \right| \hat{\epsilon}^2_{r\text{f}} \left| \psi \otimes \phi_0 \right\rangle \right) \right)^{\frac{1}{2}} \tag{9}$$

where $\mathcal{S}_{\text{sy}}$ denotes the unit sphere in the system state space $\mathcal{H}_{\text{sy}}$, as before.

Concerning the interpretation of the quantities $\Delta_{\text{ei}} A_r$, $\Delta_{\text{ef}} A_r$ the same analysis applies to them as was given for the errors characterising an approximate measurement of a single observable, in the paragraphs following Eqs. (5) and (6). In particular, we have, by analogy with Inequality (7),

$$\left(\left\langle \psi \otimes \phi_0 \right| (\hat{\alpha}_{\text{rf}} - \bar{A}_r)^2 \left| \psi \otimes \phi_0 \right\rangle \right)^{\frac{1}{2}} \leq \Delta_{\text{ei}} A_r + \Delta A_r$$

for $r = 1, \dots, n$, where $\bar{A}_r$ denotes the initial state mean, and ΔA_r denotes the initial state uncertainty, as in Inequality (7).

Even though the $\hat{A}_r$ are non-commuting, it may still happen that there exist states for which the intrinsic uncertainties ΔA_r are all small. If the retrodictive errors $\Delta_{\text{ei}} A_r$ are also small, then the above inequalities show that there is a high probability that, for each r, the recorded value of $\hat{\alpha}_r$ will be close to the initial state expectation value $\bar{A}_r$—which provides a further illustration of the sense in which the processes under discussion may be regarded as approximate joint measurements. For examples of measurement processes to which these comments apply, see Appleby [12, 13, 16, 17, 19], and Section 5 below.

3. Approximate Measurements: POVM

An approximate measurement is most conveniently analysed in terms of the corresponding POVM (positive operator valued measure). We avoided introducing this concept at the outset because we wished to establish that one can give an adequate theoretical description of approximate measurements whilst remaining wholly within the framework of the conventional theory, as it was presented by Dirac [25] and von Neumann [26]. In particular, we wished to establish that one can introduce the concept of an approximate measurement, without being thereby compelled to introduce any unconventional, unsharp or generalized observables. However, it is certainly true the concept of a POVM represents a powerful mathematical tool. Consequently, having established that it is not anything more than a tool (at least in the present context), it is appropriate to indicate how the maximal rms errors defined in Section 2 can be expressed in terms of this construct.

As in the last section, we consider a measurement of n non-commuting observables $\hat{A}_1, \dots, \hat{A}_n$ acting on the system state space $\mathcal{H}_{\mathrm{sy}}$. The system is coupled to n commuting pointer observables $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ acting on the apparatus state space $\mathcal{H}_{\mathrm{ap}}$. Let $|a_1, \dots, a_n\rangle$ be the joint eigenvector of $\hat{\alpha}_1, \dots, \hat{\alpha}_n$ with eigenvalues $a_1, \dots, a_n$ (which, for simplicity, we assume to be non-degenerate). Let $|\phi_0\rangle$ be the initial apparatus state, and let $\hat{U}$ be the unitary evolution operator describing the measurement interaction. Let $|m\rangle$ be any orthonormal basis for the system space $\mathcal{H}_{\mathrm{sy}}$. Define, for each n-tuplet $a_1, \dots, a_n$,

$$\hat{T}_{a_1,\dots,a_n} = \sum_{m,m'} \big(\langle m| \otimes \langle a_1, \dots, a_n| \big) \hat{U} \big(|m'\rangle \otimes |\phi_0\rangle \big) \, |m\rangle \langle m'| \tag{10}$$

Unlike $\hat{U}$, which acts on the product space $\mathcal{H}_{\mathrm{sy}} \otimes \mathcal{H}_{\mathrm{ap}}$, the operators $\hat{T}_{a_1,\dots,a_n}$ act just on the system space $\mathcal{H}_{\mathrm{sy}}$.

Let

$$\hat{E}_{a_1,\dots,a_n} = \hat{T}^{\dagger}_{a_1,\dots,a_n} \hat{T}_{a_1,\dots,a_n} \tag{11}$$

It is easily verified that $\hat{E}_{a_1,\dots,a_n}$ is the POVM describing the measurement outcome. In other words, the probability that, immediately after the measurement, the n pointers will be recorded as having the values $a_1, \dots, a_n$ is

$$p_{a_1,\dots,a_n} = \langle \psi | \, \hat{E}_{a_1,\dots,a_n} \, | \psi \rangle$$

where $|\psi\rangle$ is the initial state of the system, immediately before the measurement.

It is also convenient to define

$$\hat{E}^{(r)}_{a_r} = \sum_{a_1,\dots,a_{r-1},a_{r+1},\dots,a_n} \hat{E}_{a_1,\dots,a_n}$$

where the summation is over every index except for a_r. $\hat{E}^{(r)}_{a_r}$ is the POVM describing the outcome of the measurement just of $\hat{A}_r$, which is obtained by ignoring the other $n-1$ pointer readings. Thus, the probability that the r^{th} pointer reading will be a_r is given by

$$p^{(r)}_{a_r} = \langle \psi | \, \hat{E}^{(r)}_{a_r} \, | \psi \rangle$$

Starting from the definition of Eq. (8) it is not difficult to show that the r^{th} retrodictive error $\Delta_{\mathrm{ei}} A_r$ is given by

$$\Delta_{\mathrm{ei}} A_r = \left(\sup_{\psi \in \mathcal{S}_{\mathrm{sy}}} \left(\langle \psi | \sum_{a_r} (\hat{A}_r - a_r) \hat{E}^{(r)}_{a_r} (\hat{A}_r - a_r) \, | \psi \rangle \right) \right)^{\frac{1}{2}}$$

or, equivalently,

$$\Delta_{\mathrm{ei}} A_r = \left(\left\| \sum_{a_r} (\hat{A}_r - a_r) \hat{E}^{(r)}_{a_r} (\hat{A}_r - a_r) \right\| \right)^{\frac{1}{2}} \tag{12}$$

where $\| \cdot \|$ denotes the operator norm. Similarly, the r^{th} predictive error may be expressed

$$\Delta_{\mathrm{ef}} A_r = \left(\left\| \sum_{a_1, \dots, a_n} \hat{T}^{\dagger}_{a_1, \dots, a_n} (\hat{A}_r - a_r)^2 \hat{T}_{a_1, \dots, a_n} \right\| \right)^{\frac{1}{2}} \tag{13}$$

4. Approximate Measurements and "Unsharp Observables"

As we stressed earlier, the approach described in Sections 2 and 3 differs from the approach of many other authors in that we make no use of the concept of an "unsharp observable", or of what Clifton and Kent [3] refer to as a "generalized observable". In this section we discuss Uffink's [22] criticisms (also see Fleming [23]) of this way of describing joint measurements of non-commuting observables. The discussion will prove relevant in Section 8, where we consider Clifton and Kent's [3] argument "to rule out falsifications of non-contextual models based on generalized observables, represented by POV measures".

Historically, work on the application of POVM's to the theory of measurement has been strongly influenced by the fact that, from a mathematical point of view, the concept of a POVM (positive operator valued measure) is a generalization of the concept of a PVM (projection valued measure). There consequently arose the idea that, since observables of the ordinary, orthodox kind are represented by PVMs, therefore a POVM which is not also a PVM must represent an observable of a different, unorthodox kind.

If one takes such a view, then one has to suppose that what would naturally be regarded as an approximate measurement of (for example) position, is in fact a (non-approximate?) measurement of something else—unorthodox, or generalized, or unsharp position as it might be called. This way of thinking is certainly at variance with our ordinary intuitions. It would, for instance, not normally be argued that a ruler cannot be used to measure length properly so-called, but only generalized length. However, this objection is perhaps not crucial, for one does not expect quantum mechanical concepts necessarily to accord with classical intuition. Nevertheless, there are some pertinent questions regarding the interpretation of generalized observables which need to be answered if the concept is to be acceptable. As Uffink puts it: "one would naturally like to know *what* is being measured in a measurement of an unorthodox observable" (his emphasis).

The difficulty becomes particularly acute when it is approximate joint measurements of non-commuting observables which are in question. Proponents of the concept of an unsharp observable argue that, although (for example) the orthodox position and momentum observables cannot jointly be measured, there exists a different pair of unorthodox, "unsharp" observables which are jointly measurable. As Uffink points out, the problem with this approach is that, rather than solving the original problem (the problem of making a joint measurement of a pair of *orthodox* observables), it merely presents us with a solution to a new, ostensibly quite different problem (the problem of making a joint measurement of a pair of *unorthodox* observables). Advocates of the approach attempt to deal with this problem by arguing that the unorthodox observables which one actually measures are related to (are unsharp versions of) the orthodox observables which one would like to measure in such a way that, by making a (non-approximate?) measurement of the former,

one acquires approximate information regarding the latter. However, Uffink has identified some problems with this idea [22, 23].

It appears to us that the source of the difficulty lies in the concept of an unsharp observable which, at least so far as approximate measurements are concerned, adds a wholly unnecessary level of complication to the problem. In classical physics there is no need to introduce the concept of an "unsharp" quantity, and then attempt to show that, by measuring that, one gains approximate information about the ordinary quantity in which one is really interested. It turns out that there is no need to introduce such intermediate quantities in quantum physics either—as we showed in Sections 2 and 3 where (using ideas previously presented in Appleby [12, 13, 16, 17, 19]) we described approximate quantum mechanical measurements directly, without any unnecessary detours, as measurements of the self-same (orthodox) observables concerning which approximate information is sought.

As was shown in Section 3, the concept of a POVM plays an important role in our analysis. However, its role is simply that of a powerful mathematical construct, which can be used to describe the outcome of an approximate measurement. It is not taken to represent a new kind of observable, distinct from the observable one is trying approximately to measure.

It should be stressed that the above discussion only applies to approximate measurements of (orthodox) observables. In other contexts we would agree that the orthodox identification of "observable" with "self-adjoint operator" is too restrictive—as appears from the fact that, if this identification is correct, then phase and time are not observables (see, for example, Busch *et al* [27], Pegg and Barnett [28], Bužek *et al* [29], Oppenheim *et al* [30], Egusquiza and Muga [31], and references cited therein). It is also clearly true that the concept of a POVM plays an important role in the problem of arriving at a suitably extended concept of a physical observable. We only wish to point out that the question is not straightforward, and that a simple identification of the concept of a POVM with the concept of a generalized observable may be productive of confusion. Some of the pitfalls appear from the discussion in Uffink's paper. Others will appear from the discussion in Section 8.

5. Approximate Joint Measurements of the Projections $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$

We now specialise the theory presented in Sections 2 and 3 to the case which will be discussed in the next two sections, of an approximate joint measurement of the projections $(\mathbf{e}_1 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_2 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_3 \cdot \hat{\mathbf{L}})^2$ where $\hat{\mathbf{L}}$ is the angular momentum operator for a spin 1 system, and where the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are approximately, but perhaps not exactly orthogonal.

Let us start by considering an exact measurement of the single projection $\hat{P} = (\mathbf{n} \cdot \hat{\mathbf{L}})^2$, for an arbitrary unit vector $\mathbf{n}$. The system state space $\mathcal{H}_{\mathrm{sy}}$ is thus 3-dimensional. To measure $\hat{P}$ we couple the system to a single pointer observable $\hat{\alpha}$ which has the two (non-degenerate) eigenvalues 0 and 1. The apparatus state space $\mathcal{H}_{\mathrm{ap}}$ is thus 2-dimensional. Let $|0\rangle$, $|1\rangle$ be the eigenvectors of $\hat{\alpha}$ with eigenvalues 0 and 1 respectively. Let $\hat{\sigma}\colon \mathcal{H}_{\mathrm{ap}} \to \mathcal{H}_{\mathrm{ap}}$ be the operator defined by

$$\hat{\sigma}\,|0\rangle = -i\,|1\rangle \qquad\qquad \hat{\sigma}\,|1\rangle = i\,|0\rangle \tag{14}$$

Let $\hat{U}\colon \mathcal{H}_{\mathrm{sy}} \otimes \mathcal{H}_{\mathrm{ap}} \to \mathcal{H}_{\mathrm{sy}} \otimes \mathcal{H}_{\mathrm{ap}}$ be the unitary operator defined by

$$\hat{U} = \exp\left[i\frac{\pi}{2}\hat{P}\hat{\sigma}\right] = (1 - \hat{P}) + i\hat{P}\hat{\sigma}$$

Let the initial apparatus state be $|\phi_0\rangle = |0\rangle$. Then

$$\hat{U}\big(|\psi\rangle \otimes |0\rangle\big) = \begin{cases} |\psi\rangle \otimes |0\rangle & \text{if } \hat{P}\,|\psi\rangle = 0 \\ |\psi\rangle \otimes |1\rangle & \text{if } \hat{P}\,|\psi\rangle = |\psi\rangle \end{cases}$$

from which we see that $\hat{U}$ describes a completely ideal measurement of $\hat{P}$.

In order to obtain a joint measurement of the three projections $\hat{P}_1 = (\mathbf{e}_1 \cdot \hat{\mathbf{L}})^2$, $\hat{P}_2 = (\mathbf{e}_2 \cdot \hat{\mathbf{L}})^2$, $\hat{P}_3 = (\mathbf{e}_3 \cdot \hat{\mathbf{L}})^2$ we can chain together three ideal measurements of the kind just described, so that $\hat{P}_1$ is measured first, $\hat{P}_2$ second and $\hat{P}_3$ third, as illustrated in Figure 1. We then have three commuting pointer observables $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ acting on the 6-dimensional space $\mathcal{H}_{\text{ap}}$. Let $|\alpha_1, \alpha_2, \alpha_3\rangle$ be the joint eigenvector of $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}_3$ with eigenvalues $\alpha_1, \alpha_2, \alpha_3$. The unitary operator describing the measurement interaction is

$$\hat{U} = \hat{U}_3\hat{U}_2\hat{U}_1 = \big((1-\hat{P}_3) + i\hat{P}_3\hat{\sigma}_3\big)\big((1-\hat{P}_2) + i\hat{P}_2\hat{\sigma}_2\big)\big((1-\hat{P}_1) + i\hat{P}_1\hat{\sigma}_1\big) \tag{15}$$

where the operators $\hat{\sigma}_r$ are defined by

$$\begin{aligned} \hat{\sigma}_1\,|\alpha_1,\alpha_2,\alpha_3\rangle &= (-1)^{\bar{\alpha}_1} i\,|\bar{\alpha}_1,\alpha_2,\alpha_3\rangle \\ \hat{\sigma}_2\,|\alpha_1,\alpha_2,\alpha_3\rangle &= (-1)^{\bar{\alpha}_2} i\,|\alpha_1,\bar{\alpha}_2,\alpha_3\rangle \\ \hat{\sigma}_3\,|\alpha_1,\alpha_2,\alpha_3\rangle &= (-1)^{\bar{\alpha}_3} i\,|\alpha_1,\alpha_2,\bar{\alpha}_3\rangle \end{aligned}$$

and where we have employed the notation $\bar{0} = 1, \bar{1} = 0$. Referring to Eq. (10) we see that

$$\hat{T}_{\alpha_1\alpha_2\alpha_3} = \sum_{m,m'} \big(\langle m| \otimes \langle \alpha_1,\alpha_2,\alpha_3|\big)\hat{U}\big(|m'\rangle \otimes |0,0,0\rangle\big)\,|m\rangle\,\langle m'|$$

where $|m\rangle$ is any orthonormal basis for $\mathcal{H}_{\text{sy}}$. Defining $\hat{P}_r^{(0)} = 1 - \hat{P}_r$, $\hat{P}_r^{(1)} = \hat{P}_r$ this becomes

$$\hat{T}_{\alpha_1\alpha_2\alpha_3} = \hat{P}_3^{(\alpha_3)}\hat{P}_2^{(\alpha_2)}\hat{P}_1^{(\alpha_1)}$$

Using Eq. (11), and the fact that the $\hat{P}_3^{(\alpha_3)}$ are projections, the POVM describing the measurement outcome is

$$\hat{E}_{\alpha_1\alpha_2\alpha_3} = \hat{P}_1^{(\alpha_1)}\hat{P}_2^{(\alpha_2)}\hat{P}_3^{(\alpha_3)}\hat{P}_2^{(\alpha_2)}\hat{P}_1^{(\alpha_1)}$$

We may assume that the basis in $\mathbb{R}^3$ has been chosen in such a way that

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} \sin\psi \\ \cos\psi \\ 0 \end{pmatrix} \qquad \mathbf{e}_3 = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

where the angles ψ, θ (but not necessarily ϕ) are small. There is no loss of generality in assuming that the basis is right handed (since we are free to adjust the signs of the $\mathbf{e}_r$). Making this assumption, and working to second order in θ, ψ, we find (after some rather lengthy algebra)

$$\begin{aligned} \hat{E}_{111} &\approx \psi^2\,(1-\hat{L}_2^2) + \theta^2\,(1-\hat{L}_3^2) - \theta\psi\cos\phi\,\{\hat{L}_2,\hat{L}_3\} \\ \hat{E}_{110} &\approx (1-\theta^2)\,(1-\hat{L}_3^2) + \theta\psi\cos\phi\,\{\hat{L}_2,\hat{L}_3\} \\ \hat{E}_{101} &\approx (1-\theta^2\sin^2\phi-\psi^2)\,(1-\hat{L}_2^2) \\ \hat{E}_{011} &\approx (1-\theta^2\cos^2\phi-\psi^2)\,(1-\hat{L}_1^2) \\ \hat{E}_{100} &\approx \theta^2\sin^2\phi\,(1-\hat{L}_2^2) \\ \hat{E}_{010} &\approx \theta^2\cos^2\phi\,(1-\hat{L}_1^2) \\ \hat{E}_{001} &\approx \psi^2\,(1-\hat{L}_1^2) \\ \hat{E}_{000} &\approx 0 \end{aligned}$$

where $\{\hat{L}_2, \hat{L}_3\}$ denotes the anti-commutator.

$\langle \hat{E}_{\alpha_1\alpha_2\alpha_3} \rangle$ is the probability that the measurements of $\hat{P}_1$, $\hat{P}_2$, $\hat{P}_3$ will give the values α_1, α_2, α_3 respectively. If $\psi = \theta = 0$ then $\hat{E}_{\alpha_1\alpha_2\alpha_3}$ reduces to the PVM (projection valued measure)

$$\hat{E}_{110} = 1 - \hat{P}_3 \qquad \hat{E}_{101} = 1 - \hat{P}_2 \qquad \hat{E}_{011} = 1 - \hat{P}_1$$

$$\hat{E}_{111} = \hat{E}_{100} = \hat{E}_{010} = \hat{E}_{001} = \hat{E}_{001} = 0$$

and the probability of the outcome of the measurement being one of the "illegal" combinations 111, 100, 010, 001, 000 is zero. If, however, θ, ψ are not both $= 0$, then the probability of obtaining one of these combinations, though small, is not exactly zero—as was to be expected.

Using Eq. (12) we obtain (after some algebra) the following expressions for the retrodictive errors, to lowest order in ψ, θ:

$$\Delta_{\text{ei}} P_1 \approx 0 \tag{16}$$

$$\Delta_{\text{ei}} P_2 \approx |\psi| \tag{17}$$

$$\Delta_{\text{ei}} P_3 \approx |\theta| \tag{18}$$

For the sake of completeness we also give the formulae for the predictive errors, to lowest order in ψ, θ:

$$\Delta_{\text{ef}} P_1 \approx \left(2(\psi^2 + \theta^2 \cos^2 \phi)\right)^{\frac{1}{2}} \tag{19}$$

$$\Delta_{\text{ef}} P_2 \approx |\sqrt{2}\theta \sin \phi| \tag{20}$$

$$\Delta_{\text{ef}} P_3 \approx 0 \tag{21}$$

It is not difficult to see that the equalities $\Delta_{\text{ei}} P_1 = \Delta_{\text{ef}} P_3 = 0$ are actually exact. This is because the joint measurement is constructed by stringing together a sequence of measurements which are individually ideal. The errors are entirely attributable to the disturbance of the system caused by the successive measurements. The measurement of $\hat{P}_1$ comes first, there has been no preceding measurement to alter the state of the system, and so it is retrodictively ideal: which is why $\Delta_{\text{ei}} P_1 = 0$. The measurement of $\hat{P}_3$ comes last, there is no subsequent measurement to alter the state of the system, and so it is predictively ideal: which is why $\Delta_{\text{ef}} P_3 = 0$.

We have assumed that the measurements of the three operators $\hat{P}_r$ are performed sequentially, one after the other, because that is the easiest case to analyse. However, it is perfectly possible to apply these methods to cases where the three measurements are performed all at once (so to speak). For instance, one might consider the evolution described by the Hamiltonian

$$\hat{H} = \hat{H}_{\text{sy}} + \hat{H}_{\text{ap}} + \hat{H}_{\text{meas}}$$

$\hat{H}_{\text{sy}}$ and $\hat{H}_{\text{ap}}$ are the Hamiltonians describing the free evolution of the system and apparatus respectively. $\hat{H}_{\text{meas}}$ is the time-dependent Hamiltonian describing the measurement interaction, given by

$$\hat{H}_{\text{meas}} = -\frac{h}{4} f(t) \left(\hat{P}_1\hat{\sigma}_1 + \hat{P}_2\hat{\sigma}_2 + \hat{P}_3\hat{\sigma}_3\right)$$

where f is a "bump" function, which is zero outside the short time interval $[0, \tau]$, and which satisfies the normalisation condition $\int_0^\tau dt f(t) = 1$. If τ is sufficiently small then the unitary operator describing the evolution between $t = 0$ and $t = \tau$ is approximately given by

$$\hat{U}' \approx \exp\left[i\frac{\pi}{2}\left(\hat{P}_1\hat{\sigma}_1 + \hat{P}_2\hat{\sigma}_2 + \hat{P}_3\hat{\sigma}_3\right)\right]$$

If the vectors $\mathbf{e}_r$ are exactly orthonormal, then $\hat{U}'$ coincides with the operator $\hat{U}$ given by Eq. (15). Otherwise it does not. However, it is easily seen that it still describes an approximate joint measurement of the operators $\hat{P}_r$.

Every measurement occupies a finite time interval. There is no difference in principle between joint measurements which are performed sequentially, so that the measurement of each individual observable is allotted its own individual time-slice; and joint measurements which are performed contemporaneously, so that the measurement of each individual observable takes up the whole of the time which is allotted to all.

6. A Modified Kochen-Specker Argument

We now apply the concepts and methods developed in the last four sections to the questions posed in the Introduction. We consider a spin 1 system, with angular momentum $\hat{\mathbf{L}}$; and we consider the problem of making a joint measurement of the observables $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$, for some triad $\mathbf{e}_r$.

The argument in this section was originally inspired by some of the points made by Mermin [8]. However, it appears to us that our formulation is considerably sharper than Mermin's. Moreover, Mermin does not remark on the need to assume that the alignment errors are statistically independent (see below).

Before proceeding further, it will be helpful to introduce some terminology. Suppose that an analyzer is designed to measure the observable $(\mathbf{n} \cdot \hat{\mathbf{L}})^2$ but, due to the imprecision in the alignment of the analyzer, does in fact measure the slightly different observable $(\mathbf{n}' \cdot \hat{\mathbf{L}})^2$, where $\mathbf{n}$ and $\mathbf{n}'$ are both unit vectors. Then we will refer to $(\mathbf{n} \cdot \hat{\mathbf{L}})^2$ as the target observable, and to the unit vector $\mathbf{n}$ as the target alignment; while $(\mathbf{n}' \cdot \hat{\mathbf{L}})^2$ will be referred to as the actual observable, and $\mathbf{n}'$ as the actual alignment.

Kochen and Specker make no allowance for the imprecision in any real measurement procedure. They consequently assume that the measurements they consider are all ideal (in the sense explained in Section 2), and they assume that the actual observables exactly coincide with the target observables. They further assume that, in a measurement of the three observables $(\mathbf{e}_r \cdot \hat{\mathbf{L}})^2$, the vectors $\mathbf{e}_r$ can be any orthonormal triad contained in the real unit 2-sphere, S_2.

MKC, by contrast, recognise that the imprecision of any real experimental procedure means that the actual alignments $\mathbf{e}'_r$ may be slightly different from the target alignments $\mathbf{e}_r$. This permits them to make the crucial postulate, that the actual alignments are constrained to lie in a proper, dense subset $S'_2 \subset S_2$. However, MKC continue to assume that the triad $\mathbf{e}'_r$ is precisely orthonormal, and that the measurements of the observables $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$ are all ideal. As we discussed in the Introduction, these assumptions are unduly restrictive. The argument of MKC has little force unless it can be extended to the case when the triad $\mathbf{e}'_r$ is not precisely orthonormal and when, in consequence, the measurements of the (non-commuting) observables $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$ are not ideal. This is the problem we now address.

The argument which follows does not depend on any assumption regarding the specific manner in which the measurements of the observables $(\mathbf{e}'_r \cdot \hat{\mathbf{L}})^2$ are performed. The measurements could be performed sequentially, by three separate analyzers, as illustrated in Fig. 1. However, the argument applies equally well to the case when the measurements are performed contemporaneously, by a single piece of apparatus, as discussed at the end of Section 5.

In order to proceed it is necessary to make a definite hypothesis as to the distribution of actual alignments corresponding to a given target alignment. The most straightforward hypothesis, and the assumption on which the argument of this section will be based, is that the actual alignments are distributed randomly. We will

further assume that, in the case of an apparatus which is designed to measure several different target observables, the actual observables are distributed independently. In other words, we assume that, for each possible target alignment $\mathbf{n}$, there is a probability measure $\mu_{\mathbf{n}}$ defined on the set S_2' (*i.e.*, the set to which MKC postulate that the actual alignment must belong) such that, in a measurement of the target observables $(\mathbf{e}_1 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_2 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_3 \cdot \hat{\mathbf{L}})^2$, the probability that the actual alignments $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ lie in the set $A_1 \times A_2 \times A_3 \subseteq S_2' \times S_2' \times S_2'$ is $\mu_{\mathbf{e}_1}(A_1)\mu_{\mathbf{e}_2}(A_2)\mu_{\mathbf{e}_3}(A_3)$.

The probability measure $\mu_{\mathbf{n}}$ depends, not only on the vector $\mathbf{n}$, but also on the construction of the apparatus. An apparatus which was constructed differently, so as to permit the alignments to be fixed more precisely, would have a different associated probability distribution.

The argument of this section is crucially dependent on the assumption that the alignment errors are statistically independent. We will consider the hypothesis that the distributions are not independent in Section 7.

We will denote the set of possible target alignments S_2''. The question arises: what, exactly, is this set? What conditions must a vector satisfy in order to be a possible target alignment? One could argue that *every* vector $\in S_2$ is a possible target alignment. However, it might be thought that this view would be too extreme. The target observable is the observable which the analyzer is designed to measure; and it may reasonably be argued that it is possible to design an instrument to measure some observable if and only if it is possible unambiguously to describe that observable. We will accordingly take the view that S_2'', the set of possible target alignments, consists of those unit vectors $\in S_2$ which are finitely specifiable—that is, which can be specified in standard mathematical notation, by means of a string consisting of finitely many characters.

The set S_2'' so defined is countable, like the set S_2'. However, unlike the set S_2', the set S_2'' includes Kochen-Specker (KS) uncolourable sets. For instance, it includes the uncolourable set given by Kochen and Specker [4] themselves, and the one given by Peres [7, 32] (the vectors belonging to these sets manifestly are finitely specifiable, since the authors explicitly do so specify them). It follows that S_2'' is itself KS uncolourable.

The fact that S_2'' is KS uncolourable is crucial. It opens the way to a modified version of the KS theorem.

Now consider a valuation $f\colon S_2' \to \{0,1\}$. The fact that S_2' is countable means that f is automatically $\mu_{\mathbf{n}}$-measurable, for every $\mathbf{n} \in S_2''$. Consequently, we may define for each $\mathbf{n} \in S_2''$,

$$p(\mathbf{n}) = \mu_{\mathbf{n}}\big(\{\mathbf{n}' \in S_2' : f(\mathbf{n}') = 1\}\big)$$

$p(\mathbf{n})$ is the probability that, if the analyzer is designed to measure the target observable $\hat{P}_{\mathbf{n}}$, then the vector $\mathbf{n}' \in S_2'$ characterising the actual alignment of the analyzer will have f-value $= 1$.

Using the function $p(\mathbf{n})$ we next define an induced valuation $\tilde{f}\colon S_2'' \to \{0,1\}$ by

$$\tilde{f}(\mathbf{n}) = \begin{cases} 0 & \text{if } p(\mathbf{n}) < 0.5 \\ 1 & \text{if } p(\mathbf{n}) \geq 0.5 \end{cases}$$

The valuation f is defined on the set S_2', which is KS colourable. However, the induced valuation $\tilde{f}$ is defined on S_2'' which, as we have seen, is KS uncolourable. It follows that there must exist an orthonormal triad $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in S_2''$ which $\tilde{f}$-evaluates to one of the "illegal" combinations 111, 100, 010, 001, 000.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in S_2''$ be such a triad, and suppose that the sequence of three analyzers illustrated in Fig. 1 is used to make a joint measurement of the corresponding target observables, $(\mathbf{e}_1 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_2 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}_3 \cdot \hat{\mathbf{L}})^2$. Let $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3' \in S_2'$ represent the

actual alignments of the analyzers. We will assume that the precision with which the analyzers can be aligned is very high, so that the triad $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is very nearly orthonormal.

It follows from the definitions of f, $\tilde{f}$ that, for each r, there is probability ≥ 0.5 that $f(\mathbf{e}'_r) = \tilde{f}(\mathbf{e}_r)$. Consequently, there is a non-negligible probability that $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ f-evaluates to one of the "illegal" combinations 111, 100, 010, 001, 000 (in fact it is straightforward, though somewhat tedious to show that the probability of obtaining one of these combinations is ≥ 0.5). On the other hand, the fact that the triad $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ is almost orthonormal, and the results proved in Section 5, together imply that the probability that the result of the measurement will be one of these combinations is ≈ 0. It follows that there is non-negligible probability (in fact, probability $\gtrsim 0.5$) that, for at least one of the observables $(\mathbf{e}'_1 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}'_2 \cdot \hat{\mathbf{L}})^2$, $(\mathbf{e}'_3 \cdot \hat{\mathbf{L}})^2$, the measured value is not close to the f-value.

This establishes that, if the alignment errors are random, and statistically independent, then the model must exhibit a form of contextuality: for it means that the probable outcome of an approximate measurement must, in general, be strongly dependent, not only the observable which is being measured, but also on the particular way in which the measurement is carried out. It follows that, if the stated assumptions are true, then the model fails to satisfy clause 3 of the AOV principle (as stated in the Introduction).

7. The Assumption of Independence

It is easily seen that the assumption that the alignment errors are statistically independent is crucial to the argument in the last section. For instance, one can envisage a model in which the errors are correlated in such a way that the actual alignments are always orthogonal. In that case the situation would reduce to the one considered by MKC. It should, however, be noted that this assumption, besides being somewhat implausible, is empirically falsifiable. We have, until now, been following MKC in assuming that the discrepancies between target and actual alignments are a consequence of the limited precision of the measuring device. However, one can equally well consider a case where the alignment "errors" are artificially controlled, using a random number generator, to be much larger than the minimum attainable errors (and consequently measurable). The argument of the last section applies to this case just as well as to the case when the errors are due to the finite precision of the instrument. Consequently, if one wished to avoid the conclusion to that argument in the manner suggested, then one would have to postulate that it is physically impossible to set up the apparatus in such a way that the angles between the alignments are measurably different from 90°—which (quite apart from the implausibility of the suggestion) is a definite empirical prediction.

Of course, the fact that, in the MKC models, for each of the values 0 and 1, the set of vectors which are assigned that value constitute a dense subset of S_2, means that these models are very flexible. Consequently, it may be that there exist other, rather more subtle, postulates regarding the distribution of actual alignments which are not empirically falsifiable, and which do have the property that clause 3 of the AOV principle is then satisfied. However, it would not be very easy to prove that this was the case. In a *complete* hidden variables theory, the probability measure describing the distribution of actual alignments is not a feature which one is free simply to postulate. It has to be derived, from the dynamics of the interacting system+apparatus+environment composite. The models discussed by MKC are incomplete, since they do not include a specification of the dynamics. It is a highly non-trivial question, as to whether there exists a dynamics which, in every situation, gives rise to a probability distribution having the desired properties—not only in

situations where the alignment errors arise "naturally", but also in situations where the errors are adjusted "by hand" (in the manner described in the last paragraph). In the absence of a solution to this problem, the question as to whether there exist hidden variables theories satisfying clause 3 of the AOV principle must be regarded as open.

Suppose, however, that a suitable dynamics could be constructed. Regarded from the perspective of classical intuition the carefully adjusted correlations between the different alignment errors in a complex apparatus which such a model must exhibit would seem very peculiar (they would have something of the flavour of a "conspiracy"). However, what is perhaps rather more to the point is the fact that this phenomenon would itself represent a kind of contextuality: for it would mean that the statistical fluctuations in an analyzer were *ineluctably* dependent on the overall experimental context in which the analyzer was used. A theory of this kind (supposing that it could be constructed) would not so much nullify the contextuality asserted in the conclusion to the Kochen-Specker theorem, as change the locus of the contextuality, from the system, onto the alignment errors.

The conclusion consequently seems to be that, although one may modify its precise form, some kind of contextuality must appear somewhere, in any hidden variables theory.

The problem of trying to nullify the non-classical elements in a hidden variables theory might be compared with the problem of trying to nullify the rucks in a badly fitted carpet. The carpet corresponds to the hidden-variables theory. The nails holding the carpet down correspond to the empirical data. One has a certain amount of freedom to move the rucks around; and with sufficient ingenuity one may succeed in making them less noticeable. However, so long as the nails remain in place, the rucks cannot actually be eliminated.

8. Clifton and Kent's POVM Theorem

In addition to the part of their argument which we have been considering up to now, Clifton and Kent [3] also prove a theorem which, according to them, "rule[s] out falsifications of non-contextual models based on generalized observables, represented by POV measures" (also see Kent [2]). As we have seen, an approximate joint measurement of non-commuting observables is most conveniently described in terms of a POVM. Clifton and Kent's theorem 2 also concerns measurements whose outcome is described in terms of a POVM. Consequently, it may at first sight seem that their theorem 2 contradicts the result proved in Section 6 of this paper. One purpose of this section is to show that this is not in fact the case. The other purpose is to point out that, although Clifton and Kent's theorem 2 is valid if regarded as a piece of pure mathematics, its physical significance is more questionable.

Clifton and Kent show, that given any finite dimensional Hilbert space $\mathcal{H}$, there exists a set $\mathcal{A}_{\mathrm{d}}$ of positive operators acting on $\mathcal{H}$, and a truth function $t_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{d}} \to \{0,1\}$ such that

1. If $\{A_i\} \subseteq \mathcal{A}_{\mathrm{d}}$ is a finite positive operator resolution of the identity (so that $\sum_i A_i = 1$), then
$$\sum_i t_{\mathcal{A}}(A_i) = 1$$
2. the set of finite positive operator resolutions of the identity contained in $\mathcal{A}_{\mathrm{d}}$ is a countable, dense subset of the set of all finite positive operator resolutions of the identity ("dense" relative to a topology defined in their paper).

They further argue that the set of all such functions $t_{\mathcal{A}}$ is sufficiently large for the theory to be able to recover the statistical predictions of any density operator (in so far as these are testable using finite precision instruments).

In considering this claim we note, first of all, that every projection is also a positive operator. Consequently, the set $\mathcal{P}_{\mathrm{d}}$ of admissible projections which features in Clifton and Kent's theorem 1 should be contained in the set $\mathcal{A}_{\mathrm{d}}$ of admissible positive operators which features in their theorem 2. Also, for any given assignment of hidden variables, the truth function $t_{\mathcal{P}} \colon \mathcal{P}_{\mathrm{d}} \to \{0,1\}$ should be the restriction to $\mathcal{P}_{\mathrm{d}}$ of the truth function $t_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{d}} \to \{0,1\}$. It is not entirely clear from Clifton and Kent's paper that these requirements can be satisfied. For the sake of argument, we will assume that they are satisfied.

We next address the question, as to what, if any, connection exists between Clifton and Kent's theorem 2 and the result which we proved in Section 6 of this paper. It is true that both of these results concern measurements whose outcome is described by a POVM (positive operator valued measure) which is not also a PVM (projection valued measure). However, there is a significant difference between the way in which the POVM is interpreted. As we stressed in Section 4, the measurements which we consider are approximate measurements of *ordinary* observables. The role of the POVM is simply to provide a convenient mathematical description of the measurement outcome. By contrast, Clifton and Kent take it that the POVM's which feature in their theorem (and which are not also PVM's) represent an entirely new species of *generalized* observable.

Clifton and Kent do not provide any explicit details, as to how these generalized observables are to be measured. However, they appear to assume that the function $t_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{d}} \to \{0,1\}$ can be specified independently of the function $t_{\mathcal{P}} \colon \mathcal{P}_{\mathrm{d}} \to \{0,1\}$ (as we noted above, they do not even explicitly impose the requirement that $\mathcal{P}_{\mathrm{d}}$ should be contained in $\mathcal{A}_{\mathrm{d}}$, and that $t_{\mathcal{P}}$ should be the restriction of $t_{\mathcal{A}}$). This suggests that they are assuming that, corresponding to the new class of generalized observables, there exists a new class of generalized measurements.

In the approximate measurement procedures which we described in Sections 2, 3 and 5 the final result is obtained by recording the pointer positions. The primary mathematical construct describing the result of the measurement is thus the PVM which gives the distribution of pointer positions. The POVM is a secondary construct which is defined in terms of this PVM [see Eqs. (10) and (11)]. Let $\hat{E}_{a_1,\dots,a_n}$ be the element of the POVM which describes the probability that the pointer positions will be $a_1,\dots,a_n$. Then the admissibility of $\hat{E}_{a_1,\dots,a_n}$, as an operator describing the outcome of a physically possible approximate measurement process, entirely depends on the admissibility of the corresponding projection operator, acting on the apparatus state space. That is, $\hat{E}_{a_1,\dots,a_n}$ is admissible if and only if the corresponding projection belongs to the set $\mathcal{P}^{\mathrm{ap}}_{\mathrm{d}}$, of admissible apparatus projections. Moreover, if $\hat{E}_{a_1,\dots,a_n}$ is admissible, then it should be assigned the same truth value as the corresponding apparatus projection. Clifton and Kent, on the other hand, because they interpret the POVM as the mathematical representation of a completely different kind of observable, assume that they are free to fix the set $\mathcal{A}_{\mathrm{d}}$ and truth function $t_{\mathcal{A}} \colon \mathcal{A}_{\mathrm{d}} \to \{0,1\}$ without having any regard for the apparatus set $\mathcal{P}^{\mathrm{ap}}_{\mathrm{d}}$ and truth function $t^{\mathrm{ap}}_{\mathcal{P}} \colon \mathcal{P}_{\mathrm{d}} \to \{0,1\}$. Consequently, their theorem 2 has no bearing on the result proved in Section 6 of this paper.

Until now we have been concerned with the relationship (or lack of relationship) between Clifton and Kent's theorem 2 and the result which we proved in Section 6. However, similar considerations also show that there are certain obscurities regarding the significance of Clifton and Kent's result, even when it is interpreted in their

suggested terms. It is, of course, the case that not every POVM arises in the manner discussed in this paper, as a way of describing the outcome of an approximate measurement of an ordinary observable. In other contexts it may be appropriate to think of a POVM as representing a generalized observable. But, whatever the entity is called, one always needs to specify how it is to be measured. The usual answer to this question makes use of the Neumark extension theorem [7, 20, 33, 34, 35]. In the variant of this approach which was proposed by Peres [7, 35], the system of interest is combined with an ancilla, and an ideal measurement is performed on the composite. The outcome of this measurement is described by a PVM. It follows that, in this case too, once one has fixed the set $\mathcal{P}_{\mathrm{d}}$ of admissible projections for the system+ancilla composite, one is no longer free to omit the corresponding system-space positive operators from the set of admissible positive operators.

It can be seen from this that one has less freedom to choose the set $\mathcal{A}_{\mathrm{d}}$ than Clifton and Kent assume. The situation regarding the truth values which should be assigned to the members of this set is even more problematic. It is not simply that these values are already partially determined by the function $t_{\mathcal{P}}$ describing the pointer observables (in the case of an approximate measurement procedure of the kind discussed in Section 3), or the system+ancilla composite (in the case of Peres' version of the Neumark construction). It is not even clear that the truth values are determined unambiguously; for it may happen that a given POVM can be physically realized in more than one way. Suppose, for example, that the POVM described in Section 5 is alternatively realized by Peres' version of the Neumark construction. In both cases, the outcome is fixed once the relevant functions $t_{\mathcal{P}}$ are fixed; but it is far from clear that the outcomes will be the same. In short, it is questionable whether it is appropriate to think in terms of there being a well-defined truth function on the set $\mathcal{A}_{\mathrm{d}}$.

9. Conclusion

In this paper we have made a number of criticisms of the arguments of MKC. This should not be allowed to obscure the fact that their work is, in our view, both interesting and valuable. MKC have taken a question which seemed clear-cut, and shown that it is in fact much more subtle and intricate than had previously been appreciated. They have thereby significantly deepened our understanding of the conceptual implications of quantum mechanics.

Similar qualifications apply to our critical remarks (based on Uffink's [22] criticisms) concerning the use of the concept of an unsharp observable to describe approximate measurements of ordinary observables. As we stressed at the end of Section 4, these criticisms should not be taken to imply that we question the need for an extended concept of physical observable.

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